

An $\mathcal{N}=1$ Supersymmetric Coulomb Flow in IIB Supergravity

Alexei Khavaev and Nicholas P. Warner

*Department of Physics and Astronomy
and*

CIT-USC Center for Theoretical Physics

University of Southern California

Los Angeles, CA 90089-0484, USA

We find a three-parameter family of solutions to IIB supergravity that corresponds to $\mathcal{N} = 1$ supersymmetric holographic RG flows of $\mathcal{N} = 4$ supersymmetric Yang Mills theory. This family of solutions allows one to give a mass to a single chiral superfield, and to probe a two-dimensional subspace of the Coulomb branch. In particular, we examine part of the Coulomb branch of the Leigh-Strassler fixed point. We look at the infra-red asymptotics of these flows from the ten-dimensional perspective. We also make general conjectures for the lifting Ansatz of five-dimensional scalar configurations to ten-dimensional tensor gauge fields. Our solution provides a highly non-trivial test of these conjectures.

June, 2001

1. Introduction

Holographic renormalization group flows have been extensively studied via five dimension supergravity, but have only been studied to a lesser extent in the underlying ten-dimensional string theories (or in M -theory). It has become clear that the five-dimensional descriptions of such flows can prove to be a powerful tool, but that the ten-dimensional descriptions are essential to a proper understanding of the infra-red behaviour of such flows (see, for example, [1]).

There are several issues in “lifting” five-dimensional supergravity solutions to ten-dimensional supergravity. First, is the obvious fact that a ten-dimensional lift may not exist: Many of the five-dimensional solutions that are considered as holographic RG flows do not appear to be solutions of a supergravity limit of an underlying string theory. Thus, while they may represent interesting conjectures, such flows may have only a tenuous connection to the well-established holographic string dualities. There are, however, many five-dimensional supergravity theories that are connected with an underlying string compactification, and these five-dimensional supergravity theories fall into two classes: Effective low energy theories, and, consistent truncations. The former class of solutions only represent approximations to some higher-dimensional lifts, while the latter class of solutions will have exact lifts to ten dimensions. It is therefore the consistent truncations that have the best chance of allowing some reasonable interpretation of the infra-red end of the flow: One only has to contend with the supergravity approximation to string theory, and not with the possible breakdown of an effective five-dimensional description. This may suggest that knowing that a five-dimensional theory is a consistent truncation is enough to extract all possible supergravity information about the IR limit using that five-dimensional description. This is, however, not true, and it has been shown in several recent papers that it is essential to reconstruct the full ten-dimensional solutions in order to understand properly the IR asymptotics [2,3,4,1].

In this letter we will consider gauged $\mathcal{N} = 8$ supergravity in five dimensions, which is widely believed to a consistent truncation of the S^5 compactification of IIB supergravity. The dual field theory is $\mathcal{N} = 4$ supersymmetric Yang-Mills on $D3$ -branes [5]. The truncation to gauged $\mathcal{N} = 8$ supergravity represents a truncation to perturbations involving bilinear operators and associated vevs in the Yang-Mills theory on the brane. Several flows of this model have now been lifted to IIB supergravity in ten dimensions [6,3,4], but the general story is far from complete. The general formula for the ten-dimensional metric of the lift is known [7,3], but the formulae for the lifts of the tensor gauge fields are not. Our purpose here is two-fold. First, we exhibit a new ten-dimensional solution that represents

a three-parameter RG flow. Two of the parameters represent independent scalar masses or vevs, while the third represents a single fermion mass. This enables us to extend the results of [4] that describe the $\mathcal{N} = 1$ supersymmetric RG flow to the non-trivial ‘‘Leigh-Strassler’’ conformal fixed point. In particular we are able to include another Coulomb branch parameter and thus probe the Coulomb branch of the Leigh-Strassler fixed point theory. The five-dimensional description of this three parameter family of flows was considered in [8], but here we give the ten-dimensional lift, and we are able to extract the brane geometry very explicitly.

The second purpose of this letter is to give a conjecture for the general lift of all the tensor gauge fields on the internal 5-sphere in IIB supergravity. This conjecture is based upon educated guess-work: It fits all known, non-trivial lifts and provides the basic Ansatz for the solution presented here. Our new solution thus represents a highly non-trivial test of the conjecture.

We will begin in section 2 by giving the general conjectured formulae for lifting to ten dimensions. In section 3 we will summarize the results of [8] that are relevant, and we will use this and our conjectures of section 2 to generate an Ansatz for the new solution. In section 4 we will give the solution explicitly, and we will discuss its asymptotic behaviour.

2. An Ansatz for Tensor Gauge Fields

We will follow the conventions of [9] throughout. Recall the 42 scalars of the $\mathcal{N} = 8$ theory in five dimensions are parametrized by a 27×27 matrix of $E_{6(6)}$. This matrix is naturally decomposed into blocks labelled: $\mathcal{V}_{J\alpha}{}^{ab}$ and \mathcal{V}^{IJab} , while the corresponding blocks of the inverse matrix are labelled: $\tilde{\mathcal{V}}^{J\alpha}{}_{ab}$ and $\tilde{\mathcal{V}}_{IJab}$. As was argued in [7,3], the full Ansatz for the ten-dimensional metric is:

$$ds^2 = \Delta^{-\frac{2}{3}} ds_{1,4}^2 + d\hat{s}_5^2, \quad (2.1)$$

where the inverse metric of $d\hat{s}_5^2$ is given by:

$$\Delta^{-\frac{2}{3}} \hat{g}^{pq} = \frac{1}{a^2} K^{IJp} K^{KLq} \tilde{\mathcal{V}}_{IJab} \tilde{\mathcal{V}}_{KLcd} \Omega^{ac} \Omega^{bd}. \quad (2.2)$$

In this equation $K^{mIJ} = -K^{mJI}$, $I, J = 1, \dots, 6$ are the Killing fields on the S^5 , and Ω^{ab} is the $USp(8)$ symplectic invariant. The quantity, Δ , is defined by

$$\Delta \equiv \sqrt{\det(\hat{g}_{mp} \overset{\circ}{g}{}^{pq})}, \quad (2.3)$$

where the inverse metric, $\overset{\circ}{g}{}^{pq}$, is that of the “round” S^5 . The warp factor, Δ , can thus be determined by taking the determinant of both sides of (2.2).

The full Ansatz for the dilaton was proposed in [3]. That is, the IIB dilaton is represented by an $SL(2, \mathbb{R})$ matrix, S , and the gauge invariant quantity is the matrix, $\mathcal{M} = S S^T$. This matrix, \mathcal{M} , appears in the kinetic term of the 3-form field strengths. It was argued in [3] that the dilaton Ansatz is given by:

$$\Delta^{-\frac{4}{3}} \mathcal{M}_{\alpha\beta} = \text{const} \times \mathcal{V}_{I\alpha}{}^{ab} \mathcal{V}_{J\beta}{}^{cd} x^I x^J \Omega_{ac} \Omega_{bd}, \quad (2.4)$$

where the x^I are the cartesian coordinates for an \mathbb{R}^6 embedding of the compactification 5-sphere. That is, the S^5 , and its deformations are defined by the surface: $\sum_I (x^I)^2 = 1$.

The quantity, Δ , can also be determined by taking the determinant of both sides of (2.4).

The five-dimensional flows have a metric of the form:

$$ds_{1,4}^2 = dr^2 + e^{2A(r)} (\eta_{\mu\nu} dx^\mu dx^\nu), \quad (2.5)$$

In this letter we will use “mostly +” Lorentzian metrics. A supersymmetric flow can usually be characterized in terms of a superpotential, W , and this superpotential is related to the scalar matrix as follows. One defines a $USp(8)$ tensor, W_{ab} , via:

$$W_{ab} \equiv -\epsilon^{\alpha\beta} \delta^{IJ} \Omega^{cd} \mathcal{V}_{I\alpha ac} \mathcal{V}_{J\beta bd}. \quad (2.6)$$

One can often extract a superpotential from W_{ab} when the latter has a constant eigenvector. More precisely, the matrix $W_{ac} W^{bc}$ is hermitian, and symplectic invariance implies that the eigenvectors come in symplectic pairs. One can choose a $USp(8)$ gauge in which these eigenvectors are also eigenvectors of W_{ab} , and if the eigenvectors are constant then the eigenvalue, W , is often a superpotential [10,8]. A five-dimensional supersymmetric flow is then given by:

$$\frac{d\varphi_j}{dr} = \frac{1}{L} \frac{\partial W}{\partial \varphi_j}, \quad \frac{dA}{dr} = -\frac{2}{3L} W, \quad (2.7)$$

where L is the radius of the AdS_5 at infinity, and the φ_j are canonically normalized scalars with kinetic term $-\frac{1}{2} \sum_j (\partial \varphi_j)^2$.

The basic philosophy behind our conjectured Ansatz is to identify the indices I, J, \dots on \mathcal{V} as indices on \mathbb{R}^6 and think of x^I as the unit normal to the deformed S^5 and then

look for building blocks that can be built out of \mathcal{V} . The first step is to create a set of *geometric* W -tensors, and in particular, define:

$$\widetilde{W}_{ab} \equiv -\epsilon^{\alpha\beta} x^I x^J \Omega^{cd} \mathcal{V}_{I\alpha ac} \mathcal{V}_{J\beta bd}. \quad (2.8)$$

In this definition we have replaced the \mathbb{R}^6 metric, δ^{IJ} , in (2.6) by the outer product of the unit normals, $x^I x^J$. Now suppose that η^a and ζ^a are two (constant) eigenvalues of W_{ab} with an eigenvalue W that represents a superpotential. Normalize these vectors to have unit length, and consider:

$$\widetilde{W} \equiv \frac{1}{2} W_{ab} (\eta^a \eta^b + \zeta^a \zeta^b). \quad (2.9)$$

We will refer to \widetilde{W} as the *geometric superpotential*. Note that it contains more information than just the superpotential, indeed the superpotential can be obtained from \widetilde{W} via:

$$W = \frac{1}{2} \sum_{I=1}^6 \frac{\partial^2}{\partial x^I \partial x^I} \widetilde{W}.$$

It is an empirical fact that in all the known lifts of flows to IIB supergravity one has:

$$A_{\mu\nu\rho\sigma}^{(4)} = \widetilde{W} e^{4A(r)} \epsilon_{\mu\nu\rho\sigma}. \quad (2.10)$$

Note that this expression is simply \widetilde{W} times the volume form on the $D3$ -brane measured using the five-dimensional metric, (2.5). While we have a heuristic justification of this formula, the primary argument in its favour is that it fits all the lifted flows given in [11,6,3,4].

In general, the Ansatz for the metric and other fields only depends upon the matrix, \mathcal{V} , whereas (2.10) explicitly depends upon the supersymmetry eigenvectors, η^a and ζ^a , and implicitly upon a choice of superpotential. Therefore we have only made an Ansatz appropriate to supersymmetric flows. On the other hand, it is possible that the particular form of (2.10) is a convenient gauge choice for $A^{(4)}$, and that there is a more general formula for the field strength, $F^{(5)}$, in terms of a geometric analog of the full supergravity potential. Certainly the radial derivative of (2.10) generates precisely the sort of terms one would need if such a conjecture were true. There is also an obvious geometric generalization of the tensor, W_{abcd} , of five-dimensional supergravity, and presumably this would be a crucial ingredient in a geometrization of the supergravity potential. We will, however, not pursue this here since (2.10) will suffice for our purposes.

To give the Ansatz for the 2-form gauge fields we need to introduce intrinsic coordinates, ξ^j , $j = 1, \dots, 5$ on the S^5 . The partial derivatives, $\frac{\partial x^J}{\partial \xi^j}$, then act as projectors from \mathbb{R}^6 onto S^5 . Consider the tensors:

$$B^\alpha{}_{ij} = k L^2 \mathcal{M}^{\alpha\beta} (x^K \mathcal{V}_{K\beta}{}^{ab}) \left(\mathcal{V}_{IJab} \frac{\partial x^I}{\partial \xi^i} \frac{\partial x^J}{\partial \xi^j} \right), \quad (2.11)$$

where $\mathcal{M}^{\alpha\beta}$ is the inverse of $\mathcal{M}_{\alpha\beta}$ defined in (2.4). The constant, k , is a dimensionless normalization constant. For $\alpha = 1, 2$ this formula yields 2-forms on S^5 , these fields transform in an $SL(2, \mathbb{R})$ doublet, and at the linearized level the formula produces exactly the correct answer for the lowest modes of the *IIB* supergravity B -fields. Indeed, we find that the foregoing formula exactly reproduces all of the B -fields (to all orders) for all of the ten-dimensional lifted solutions obtained in [3,4]¹.

We should stress that we have not proven that these formulae are correct, we have merely constructed some moderately obvious tensors from the five-dimensional scalar matrix, \mathcal{V} , and we have checked that these formulae miraculously reproduce some the rather complicated results of several known, explicit solutions of *IIB* supergravity. It is possible that these relatively simple formulae are a consequence of the special subclasses of scalars that have been considered in [11,3,4], and that some modification will be needed in general. However, we will succumb to the obvious temptation, and conjecture that (2.10) and (2.11) provide the exact lift of the five-dimensional scalar fields to the tensor gauge fields.

3. The $\mathcal{N} = 1$ Coulomb branch flows

The flows that we consider are those that involve the operators:

$$\begin{aligned} \mathcal{O}_1 &\equiv \text{Tr}(-X_1^2 - X_2^2 - X_3^2 - X_4^2 + 2X_5^2 + 2X_6^2), \\ \mathcal{O}_2 &\equiv \text{Tr}(X_1^2 + X_2^2 - X_3^2 - X_4^2), \\ \mathcal{O}_3 &\equiv \text{Tr}(\lambda_4 \lambda_4) + h.c. \end{aligned} \quad (3.1)$$

It should also be remembered that the operator:

$$\mathcal{O}_0 \equiv \text{Tr} \left(\sum_{i=1}^6 X_i^2 \right), \quad (3.2)$$

¹ There are some typographical errors in [3] and [4]. In [3] the coefficient function a_3 for the $\mathcal{N} = 2$ flow should have its sign reversed to give $a_3 = -\frac{4}{g^2} \frac{\sinh(2\chi)}{X_2} \sin \theta \cos^2 \theta$, while for the $\mathcal{N} = 1$ flow in [4] the coefficient function, a_1 , should be multiplied by i to give $a_1 = \frac{2i}{g^2} \tanh(\chi) \cos \theta$.

has no supergravity dual in the gauged $\mathcal{N} = 8$ supergravity theory, but that the field theory on the brane always adds an appropriate amount of \mathcal{O}_0 to the operators $\mathcal{O}_j, j = 1, 2$ so as to preserve supersymmetry and positivity.

We will denote the supergravity scalars dual to these operators as α, β and χ respectively. We introduce $\rho \equiv e^\alpha, \nu \equiv e^\beta$, and define: $\varphi_1 = \frac{1}{\sqrt{6}}\alpha, \varphi_2 = \frac{1}{\sqrt{2}}\beta$ and $\varphi_3 = \chi$ and note that the φ_j are canonically normalized scalars with kinetic term $-\frac{1}{2} \sum_j (\partial\varphi_j)^2$.

A superpotential for this flow was given in [8]:

$$W = \frac{1}{4} \rho^4 (\cosh(2\chi) - 3) - \frac{1}{4\rho^2} (\nu^2 + \nu^{-2}) (\cosh(2\chi) + 1). \quad (3.3)$$

We have replaced $\alpha \rightarrow -\alpha$ in the formula of [8] so as to bring it into line with earlier papers like [10,3,4]. The equations of motion are:

$$\frac{d\alpha}{dr} = \frac{1}{6L} \frac{\partial W}{\partial \alpha}, \quad \frac{d\beta}{dr} = \frac{1}{2L} \frac{\partial W}{\partial \beta}, \quad \frac{d\chi}{dr} = \frac{1}{L} \frac{\partial W}{\partial \chi}. \quad (3.4)$$

It was shown in [8] that this superpotential describes a family of $\mathcal{N} = 1$ supersymmetric flows in which the chiral superfield, Φ_3 is given a mass. As is familiar from [10] the flow of α and χ only describes a pure mass term for Φ_3 for a specific choice of initial velocities. The flow then runs to the non-trivial critical point of [7]. More generally, with other choices of the initial velocities, and with $\beta \neq 0$ the flows can go to an unphysical regime ($\alpha \rightarrow -\infty$), or to a region ($\alpha \rightarrow +\infty$) in which the flow approaches the Coulomb branch of the original $\mathcal{N} = 4$ region – the fermion mass is swamped by the values of the vevs. In between these flows was an interesting pair of “ridge-line” flows that started at the non-trivial fixed point. These flows correspond to Coulomb branch flows from the Leigh-Strassler fixed point with non-zero vevs for either the scalars in $\text{Tr}(\Phi_1 \bar{\Phi}_1)$ or $\text{Tr}(\Phi_2 \bar{\Phi}_2)$.

The asymptotic behaviour of the supergravity scalars along these flows is:

(i) Unphysical Flow:

$$\alpha \sim \frac{1}{20} \log(\frac{5}{3} r), \quad \beta \sim \pm 3 \alpha, \quad \chi \sim -6 \alpha, \quad A \sim 2 \alpha \sim \frac{1}{10} \log(\frac{5}{3} r), \quad (3.5)$$

(ii) Generic asymptotic flow to the $\mathcal{N} = 4$ Coulomb branch:

$$\chi \rightarrow a r^{\frac{3}{4}} \rightarrow 0, \quad \alpha \sim -\frac{1}{4} \log(\frac{4}{3} r), \quad \beta \rightarrow \beta_0, \quad A \sim \frac{1}{4} \log(r), \quad (3.6)$$

(iii) Ridge-line flow:

$$\begin{aligned}\alpha &\sim -\frac{1}{4} \log\left(\frac{2}{3}r\right), & \beta &\sim \pm(3\alpha - \chi^2), \\ \chi^2 &\sim \frac{1}{a-6 \log(r)}, & A(r) &\sim \log(r).\end{aligned}\tag{3.7}$$

It is useful to note that the flows with $\chi = 0, \beta = 0$ and $\chi = 0, \beta = \pm 3\alpha$ represent three completely equivalent $SO(4) \times SO(2)$ invariant Coulomb branch flows. These three different flows are simply discrete $SO(6)$ rotations of one another.

The fermion mass parameter vanishes asymptotically in both the physical flows, which is consistent with the dominance of a Coulomb parameters in the infra-red. However, the vanishing of χ occurs much more slowly along the ridge-line flow. Moreover, the five-dimensional geometry behaves very differently because of the different asymptotics of $A(r)$. To gain further insight into the holographic description of these flows, we will examine and contrast them from the ten-dimensional perspective.

4. The solution to IIB supergravity

We now use the formulae of section 2 to obtain Ansätze for the metric and tensor gauge fields for general values of α, β and χ . We will parametrize the 5-sphere in \mathbb{R}^6 by taking:

$$\begin{aligned}u_1 &\equiv x_1 + i x_2 = \cos \theta \cos \phi e^{i \varphi_1}, & u_2 &\equiv x_3 - i x_4 = \cos \theta \sin \phi e^{-i \varphi_2}, \\ u_3 &\equiv x_5 - i x_6 = \sin \theta e^{-i \varphi_3}.\end{aligned}\tag{4.1}$$

The ten-dimensional metric is then given by:

$$ds_{10}^2 = \Omega^2 ds_{1,4}^2 + ds_5^2,\tag{4.2}$$

where Ω , is given by:

$$\Omega \equiv \Delta^{-\frac{1}{3}} = (\cosh \chi)^{\frac{1}{2}} \left(\rho^{-2} (\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi) \cos^2 \theta + \rho^4 \sin^2 \theta \right)^{\frac{1}{4}},\tag{4.3}$$

and the metric ds_5^2 is the following metric on the deformed S^5 :

$$\begin{aligned}ds_5^2 &= L^2 \Omega^{-2} \left[\rho^{-4} \left(\cos^2 \theta + \rho^6 \sin^2 \theta (\nu^{-2} \cos^2 \phi + \nu^2 \sin^2 \phi) \right) d\theta^2 \right. \\ &\quad + \rho^2 \cos^2 \theta (\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi) d\phi^2 \\ &\quad - 2 \rho^2 (\nu^2 - \nu^{-2}) \sin \theta \cos \theta \sin \phi \cos \phi d\theta d\phi \\ &\quad \left. + \rho^2 \cos^2 \theta (\nu^{-2} \cos^2 \phi d\varphi_1^2 + \nu^2 \sin^2 \phi d\varphi_2^2) + \rho^{-4} \sin^2 \theta d\varphi_3^2 \right] \\ &\quad + L^2 \Omega^{-6} \sinh^2 \chi \cosh^2 \chi \left(\cos^2 \theta (\cos^2 \phi d\varphi_1 - \sin^2 \phi d\varphi_2) - \sin^2 \theta d\varphi_3 \right)^2.\end{aligned}\tag{4.4}$$

where L is the radius of the round sphere.

This form of the metric is very natural. Recall that for $\chi = 0$ this metric must describe a Coulomb branch flow [11,12], and this in turn must be related to the extremal $D3$ -brane solutions of [12]. Correcting a minor error in [12], and replacing their radial coordinate by μ , the metric of the extremal branes is given by:

$$\begin{aligned}
ds^2 = & H_{D3}^{-\frac{1}{2}} \left[-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + H_{D3}^{\frac{1}{2}} f_{D3}^{-1} \frac{d\mu^2}{\prod_{i=1}^3 \left(1 + \frac{\ell_i^2}{\mu^2}\right)} \\
& + H_{D3}^{\frac{1}{2}} \mu^2 \left[\left(1 + \frac{\ell_1^2 \cos^2 \theta}{\mu^2} + \frac{\ell_2^2 \sin^2 \theta \sin^2 \phi}{\mu^2} + \frac{\ell_3^2 \sin^2 \theta \cos^2 \phi}{\mu^2}\right) d\theta^2 \right. \\
& + \left(1 + \frac{\ell_2^2 \cos^2 \phi}{\mu^2} + \frac{\ell_3^2 \sin^2 \phi}{\mu^2}\right) \cos^2 \theta d\phi^2 \\
& - 2 \frac{\ell_2^2 - \ell_3^2}{\mu^2} \cos \theta \sin \theta \cos \phi \sin \phi d\theta d\phi + \left(1 + \frac{\ell_1^2}{\mu^2}\right) \sin^2 \theta d\varphi_1^2 \\
& \left. + \left(1 + \frac{\ell_2^2}{\mu^2}\right) \cos^2 \theta \sin^2 \phi d\varphi_2^2 + \left(1 + \frac{\ell_3^2}{\mu^2}\right) \cos^2 \theta \cos^2 \phi d\varphi_3^2 \right].
\end{aligned} \tag{4.5}$$

where:

$$\begin{aligned}
H_{D3} &= 1 + f_D \frac{L^4}{\mu^4}, \\
f_{D3}^{-1} &= \left(\frac{\sin^2 \theta}{1 + \frac{\ell_1^2}{\mu^2}} + \frac{\cos^2 \theta \sin^2 \phi}{1 + \frac{\ell_2^2}{\mu^2}} + \frac{\cos^2 \theta \cos^2 \phi}{1 + \frac{\ell_3^2}{\mu^2}} \right) \prod_{i=1}^3 \left(1 + \frac{\ell_i^2}{\mu^2}\right).
\end{aligned} \tag{4.6}$$

As usual, in the near-brane limit where $L \gg (\mu^2 + \ell_j^2)^{\frac{1}{2}}$, we “drop the 1” in H_{D3} .

While (4.5) apparently depends upon three parameters, this is a fake in the near brane limit: If $\ell_1 = \ell_2 = \ell_3$ then the metric is still $AdS_5 \times S^5$, but merely written with a non-standard radial coordinate. It was a consequence of the arguments in [11] that this metric may then be mapped precisely onto (4.2), and the detail of the mapping were given in [13]. In particular, one has:

$$\begin{aligned}
\rho &= \left(\frac{p_2 p_3}{p_1^2} \right)^{\frac{1}{12}}, \quad \nu = \left(\frac{p_2}{p_3} \right)^{\frac{1}{4}}, \quad \Omega = (p_1 p_2 p_3)^{-\frac{1}{6}} f_D^{-\frac{1}{4}}, \\
e^{2A(r)} &= \frac{\mu^2}{L^2} (p_1 p_2 p_3)^{\frac{1}{3}}, \quad \frac{dr}{d\mu} = \frac{L}{\mu} (p_1 p_2 p_3)^{-\frac{1}{3}}; \quad p_j \equiv \left(1 + \frac{\ell_j}{\mu^2}\right).
\end{aligned} \tag{4.7}$$

The parameters, ℓ_j thus represent the initial data of α and β at infinity.

Observe that the metric (4.2) with $\chi \neq 0$ has the form of the Coulomb branch metric but with an extra factor of $\cosh \chi$ in the warp factor. Also note that the last term in (4.4) may be written in terms of the frame on the Hopf fiber:

$$L^2 \Omega^{-6} \left(\mathcal{I}m \left(u_1 d\bar{u}_1 + u_2 d\bar{u}_2 + u_3 d\bar{u}_3 \right) \right)^2. \quad (4.8)$$

Thus the metric here is a straightforward generalization of that of [4]: The Coulomb branch metric is squashed by a factor of $\cosh(\chi)$ while the Hopf fiber is stretched by a factor proportional to $\tanh(\chi)$.

The geometric superpotential, \widetilde{W} , is given by evaluating (2.9), and from this we obtain:

$$\widetilde{W} = -\frac{1}{8} \rho^{-2} (1 + \cosh(2\chi)) \cos^2 \theta (\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi) + \frac{1}{8} \rho^4 (\cosh(2\chi) - 3) \sin^2 \theta. \quad (4.9)$$

As noted in [8], the dilaton and axion are constant upon these flows, and so $\mathcal{M}_{\alpha\beta} = \delta_{\alpha\beta}$.

The Ansatz, (2.11) yields the following results for the tensor gauge fields. Let $\mathcal{B}_{\mu\nu} = B_{\mu\nu}^1 + iB_{\mu\nu}^2$, then:

$$\begin{aligned} \mathcal{B} &\equiv \frac{1}{2} \mathcal{B}_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} \Omega^{-4} L^2 \sinh(2\chi) \left(\rho^4 u_3 du_1 \wedge du_2 + \rho^{-2} \nu^2 u_1 du_2 \wedge du_3 + \rho^{-2} \nu^{-2} u_2 du_3 \wedge du_1 \right). \end{aligned} \quad (4.10)$$

It is a tedious, but straightforward exercise to verify that (4.2), (2.10) with (4.9) and (4.10), do indeed satisfy the equations of motion of *IIB* supergravity. To verify this one must, of course, use the equations of motion, (2.7) and (3.4), with the superpotential (3.3).

Thus our conjectured general consistent truncation Ansatz has passed a very non-trivial test, and we have a three parameter family of holographic RG flows in ten dimensions.

5. Infra-red limits and brane probes

The supergravity solution presented here contains the $\mathcal{N} = 1$ flows already discussed in [4]. Indeed, one can verify that if one sets $\beta = 0$, or $\nu = 1$, in all the equations in the previous section, one does indeed recover the solution of [4]. We will therefore not dwell upon these aspects of our solution, but consider the new aspects associated with the flows in β .

The physical flows identified in section 3 both have $\chi \rightarrow 0$, but at very different rates. It is evident from (4.2), (4.3) and (4.4) that if $\chi \rightarrow 0$ then the metric limits directly to the extremal rotating brane metric associated with the Coulomb branch of the $\mathcal{N} = 4$ theory [12,11]. Thus, in the infra-red, all of these $\mathcal{N} = 1$ Coulomb branch flows approach the $\mathcal{N} = 4$ Coulomb branch flows, which suggests that the mass that has been turned on for Φ_3 is ultimately swamped by the Coulomb vevs, and the model retains knowledge of its $\mathcal{N} = 4$ structure.

To understand the roles of generic flows, (ii), and the ridge-line flows, (iii), it is instructive to consider their detailed asymptotics. Recall that the $SO(2) \times SO(2) \times SO(2)$ invariant Coulomb branch flows ($\chi = 0$) are sourced by an ellipsoidal distribution of $D3$ -branes, with semi-major axes determined by the ℓ_j . Indeed, ℓ_1 is the semi-major axis in the (x_5, x_6) direction, while ℓ_2 is the semi-major axis in the (x_3, x_4) direction and ℓ_3 is the semi-major axis in the (x_1, x_2) direction².

Using the correspondence (4.7) it is thus easy to determine the asymptotic limits of the physical flows:

- a) If $\rho \rightarrow \infty$, $\nu \rightarrow \nu_0$ then $\ell_1 = 0$, but $\nu_0^2 = \frac{\ell_2}{\ell_3}$. The distribution of branes is thus an ellipsoidal shell in (x_1, x_2, x_3, x_4) with a δ -function in the other two directions. In this limit one also has:

$$\mathcal{B} \sim \frac{1}{2} \Omega^{-4} L^2 \rho^4 \sinh(2\chi) u_3 du_1 \wedge du_2 .$$

Thus B^{NS} and B^{RR} are both parallel to the brane distribution.

- b) If $\rho \rightarrow \infty$, $\nu \sim \rho^{-3}$ then $\ell_1 = \ell_2 = 0$, and the distribution of branes is a disk in the (x_1, x_2) direction. In this limit one also has:

$$\mathcal{B} \sim \frac{1}{2} \Omega^{-4} L^2 \rho^4 \sinh(2\chi) du_1 \wedge (u_3 du_2 - u_2 du_3) .$$

² To make this correspondence more precise one must make a change of variables to coordinates, y_a , defined in [12].

Thus B^{NS} and B^{RR} each have “a leg” in the brane distribution, and a leg perpendicular to it.

- c) The flow with $\rho \rightarrow \infty$, $\nu \sim \rho^3$ is the same as in b), but with $\ell_1 = \ell_3 = 0$ and with u_1 and u_2 interchanged.

The foregoing asymptotic behaviour is completely consistent with the field theory interpretation proposed in [8]. First, the superfield, Φ_3 has been given a mass, and so the only remaining fields that can receive vevs are Φ_1 and Φ_2 , and these correspond to spreading the branes in the u_1 and u_2 directions respectively. The “generic” flow limits to a two parameter family of flows that reflect the initial conditions of α and β , and correspond to the different possible scales of the vevs in Φ_1 and Φ_2 . Thus the ridge-line flows emerge as natural boundaries of the “generic” flows: either ℓ_2 or ℓ_3 vanishes, collapsing the ellipsoidal shell to a disk. The “generic” flow towards the $\mathcal{N} = 4$ Coulomb branch also washes out the \mathcal{B} -field much more rapidly than the ridge-line flow.

The foregoing picture focusses closely upon the Coulomb branch structure of the theory. It should be remembered that there is a non-trivial critical point corresponding to the Leigh-Strassler fixed point theory. This fixed point theory is conformal, and at the fixed point, the field, Φ_3 , has been “integrated out.” Moreover, while there are certainly pure Coulomb branch flows with $\beta \sim \pm 3\alpha$, the ridge-line flow, (3.7) only seems accessible from this non-trivial fixed point. More general physical flows from this fixed point are of the form (3.6) and they wash out the \mathcal{B} -field much more rapidly. We find it intriguing that the Coulomb branch flows of the Leigh-Strassler point with Φ_1, Φ_2 both non-zero rapidly flow towards the $\mathcal{N} = 4$ Coulomb branch, but that the flows with only $\Phi_1 \neq 0$ or $\Phi_2 \neq 0$ appear to be privileged in that the \mathcal{B} field vanishes far more slowly. It would be very interesting to understand this more deeply from the perspective of the physics on the brane.

Finally, we have also performed the brane probe calculation for the supergravity solution presented here, and the results do not differ significantly from those of [14]. This is not very surprising since we are generalizing the result by adding another Coulomb branch parameter. We find that potential felt by the brane probes is given by:

$$V = e^{4A(r)} (\Omega^4 - 4\widetilde{W}) = e^{4A(r)} \rho^4 (\cosh(2\chi) - 1) \sin^2 \theta, \quad (5.1)$$

which is exactly the same as was found in [14]. The potential vanishes for $\theta = 0$, and $D3$ -brane probes have the following metric on the 4-dimensional moduli space transverse to the branes:

$$ds^2 = \frac{1}{2} \tau_3 e^{2A} \left[\zeta (\rho^{-2} \cosh^2(\chi) dr^2 + L^2 \rho^2 d\phi^2) + L^2 \rho^2 (\nu^{-2} \cos^2 \phi d\varphi_1^2 + \nu^2 \sin^2 \phi d\varphi_2^2) + L^2 \rho^2 \sinh^2(\chi) \zeta^{-1} (\cos^2 \phi d\varphi_1 - \sin^2 \phi d\varphi_2)^2 \right], \quad (5.2)$$

where

$$\zeta \equiv (\nu^2 \cos^2 \phi + \nu^{-2} \sin^2 \phi). \quad (5.3)$$

This is essentially an ellipsoidally squashed version of the metric obtained in [14].

Acknowledgements

We would like to thank K. Pilch for helpful conversations. This work was supported in part by funds provided by the DOE under grant number DE-FG03-84ER-40168.

References

- [1] N.P. Warner, *Holographic Renormalization Group Flows: The View from Ten Dimensions*, Talk presented at the Second Gürsey Memorial Conference, to appear in the proceedings; **hep-th/0011207**.
- [2] J. Polchinski and M. J. Strassler, *The String Dual of a Confining Four-Dimensional Gauge Theory*, **hep-th/0003136**.
- [3] K. Pilch and N.P. Warner, *$\mathcal{N} = 2$ Supersymmetric RG Flows and the IIB Dilaton*, Nucl. Phys. **B594** (2001) 209; **hep-th/0004063**.
- [4] K. Pilch and N.P. Warner, *$\mathcal{N} = 1$ Supersymmetric Renormalization Group Flows from IIB Supergravity*, CITUSC/00-18, USC-00/02; **hep-th/0004063**.
- [5] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 **hep-th/9711200**.
- [6] K. Pilch and N.P. Warner, *A New Supersymmetric Compactification of Chiral IIB Supergravity*, Phys. Lett. **487B** (2000) 22; **hep-th/0002192**.
- [7] A. Khavaev, K. Pilch and N.P. Warner, *New Vacua of Gauged $\mathcal{N} = 8$ Supergravity in Five Dimensions*, Phys. Lett. **487B** (2000) 14; **hep-th/9812035**.
- [8] A. Khavaev and N.P. Warner, *A Class of $\mathcal{N} = 1$ Supersymmetric RG Flows from Five-dimensional $\mathcal{N} = 8$ Supergravity*, Phys. Lett. **495B** (2000) 215; **hep-th/0009159**.

- [9] M. Günaydin, L.J. Romans and N.P. Warner, *Gauged $N = 8$ Supergravity in Five Dimensions*, Phys. Lett. **154B** (1985) 268; *Compact and Non-Compact Gauged Supergravity Theories in Five Dimensions*, Nucl. Phys. **B272** (1986) 598.
- [10] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, *Renormalization Group Flows from Holography—Supersymmetry and a c -Theorem*, Adv. Theor. Math. Phys. **3** (1999) 363; [hep-th/9904017](#)
- [11] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, *Continuous Distribution of D3-branes and Gauged Supergravity*, JHEP **7** (2000) 38; [hep-th/9906194](#).
- [12] P. Kraus, F. Larsen and S. P. Trivedi, *The Coulomb branch of gauge theory from rotating branes*, JHEP **9903** (1999) 003; [hep-th/9811120](#).
- [13] I. Bakas and K. Sfetsos, *States and curves of five-dimensional gauged supergravity*, Nucl. Phys. **B573** (2000) 768; [hep-th/9909041](#).
- [14] C. V. Johnson, K. J. Lovis and D. C. Page, *Probing some $N = 1$ AdS/CFT RG flows*, JHEP **0105** (2001) 036; [hep-th/0011166](#).